Entangled radicals and Galois groups

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Let K be a field of characteristic 0.

Definition. A field extension $K \subset L$ is a radical extension if there is a subgroup $B \subset L^*$ such that

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- B/K^* is torsion;
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and $[L : K] = [B : K^*]$, and fields between K and L correspond $1-1$ to groups between K^* and B.

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Next talk: There is a polynomial time algorithm that given $d \geq 1$ and $x_1, \ldots, x_n \in \mathbb{Q}$, computes the degree of $\mathbb{Q}(\mu_d, \sqrt[d]{x_1}, \ldots, \sqrt[d]{x_n})$

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$$\begin{aligned} 5 \mid [\mathbb{F}_p^* : \langle 5 \rangle] &\implies 5 \mid \#\mathbb{F}_p^* \\ &\implies \zeta^5 = 1, \zeta \neq 1, \text{for some } \zeta \in \mathbb{F}_p, \\ &\implies 2 \mid [\mathbb{F}_p^* : \langle 5 \rangle] \end{aligned}$$

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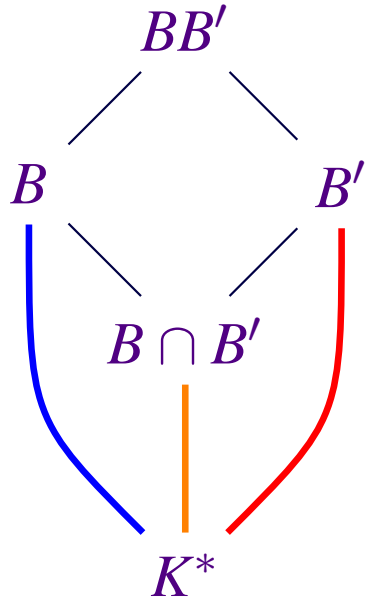
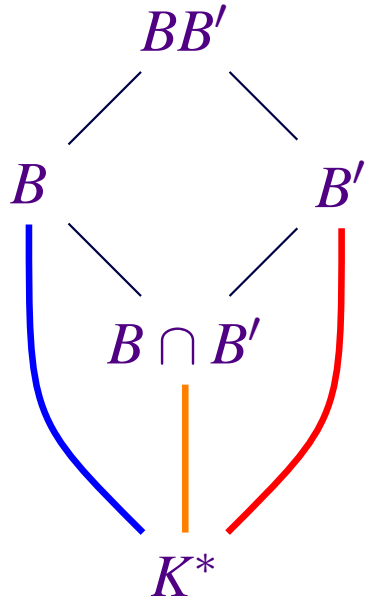
Proposition. For radical groups $B \subset C$ over K which are both Galois, the sequence

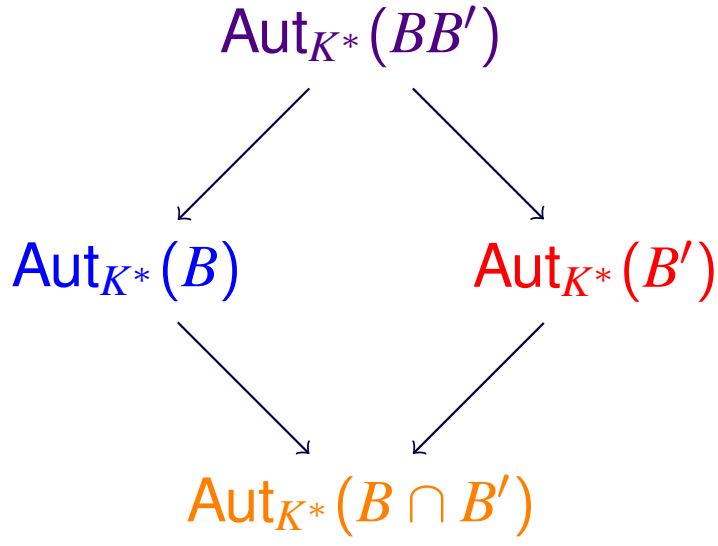
$$0 \to \mathrm{Aut}_B(C) \to \mathrm{Aut}_{K^*}(C) \to \mathrm{Aut}_{K^*}(B) \to 0$$

is an exact sequence of profinite groups.



Proposition. Suppose that $B, B' \subset C$ are radical groups over K which are all Galois. Then the groups BB' and $B \cap B'$ are also Galois over K





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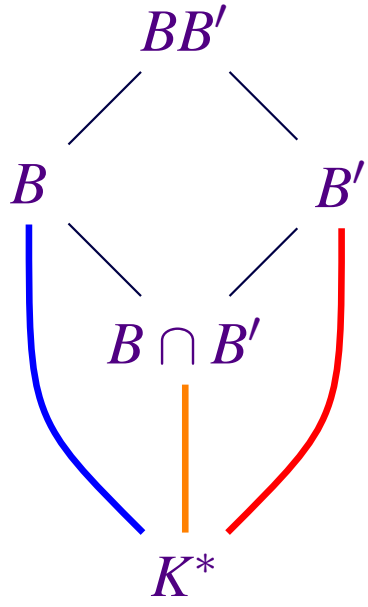

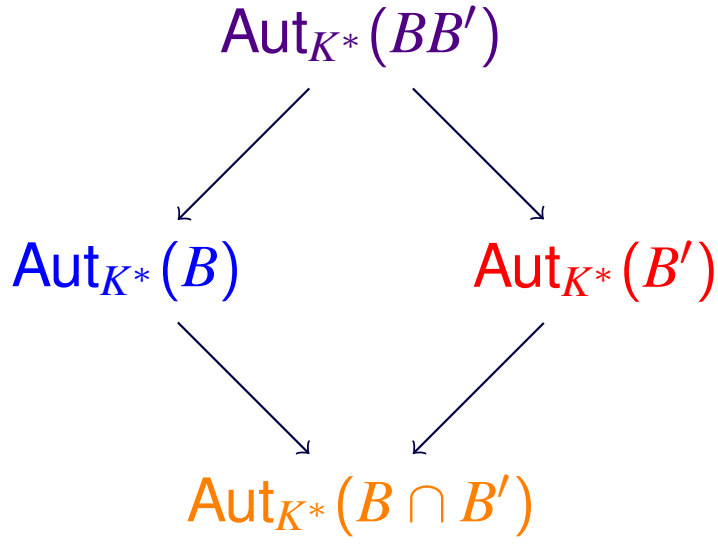

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$$\mathrm{Aut}_{K^*}(BB') = \blacksquare \times_{\blacksquare} \blacksquare.$$

Suppose B is a radical group over K which is Galois.
In two easy cases $\mathrm{Aut}_{K^*}(B)$ is abelian:

Case 1: we say B is Kummer if for all $x \in B$ there is an $n \geq 1$ so that $\mu_n \subset K^*$ and $b^n \in K^*$.

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Case 2: we say B is cyclotomic if $B = B_{\mathrm{tor}}K^*$. If $B_{\mathrm{tor}} = \mu_n$ with n finite, and $\mu_w = \mu_n \cap K^*$ then

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Corollary. If $n = [B : K^*]$ is finite then

$$\#\mathrm{Aut}_{K^*}(B) = n \prod_{\substack{\mu_p \subset B \\ \mu_p \not\subset K^*}} \frac{p-1}{p}$$

» The field “given” by a radical group «

Given a radical group B over K, and a K^*-embedding

$$\sigma : \ B \to \overline{K}^*$$

we can consider the field extension $L = K(\sigma B)$ of K.

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Proposition. The field degree $[L : K]$ does not depend on σ.

Theorem 1. Let B be a radical group over K which is Galois. Then there is an abelian profinite quotient $E(B)$ of $\mathrm{Aut}_{K^*}(B)$ so that for every embedding $\sigma : \ B \to \overline{K}^*$ the sequence

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To prove the theorem we may fix σ, and take $B \subset \overline{K}^*$.

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It follows that $E(B) \cong \mathrm{Gal}(\mathbb{Q}(\mu_p) \cap K/\mathbb{Q})$.

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Proof. Put $A = B_{\mathrm{ab}}$, then we have an exact sequence

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0 & \longrightarrow & \mathrm{Aut}_A(B) & \longrightarrow & \mathrm{Aut}_{K^*}(B) & \longrightarrow & \mathrm{Aut}_{K^*}(A) & \longrightarrow & 0 \\
 & & \uparrow f & & \uparrow g & & \uparrow h & & \\
0 & \longrightarrow & \mathrm{Gal}(\frac{K(B)}{K(A)}) & \longrightarrow & \mathrm{Gal}(\frac{K(B)}{K}) & \longrightarrow & \mathrm{Gal}(\frac{K(A)}{K}) & \longrightarrow & 0
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By Kummer theory the image of f is $\mathrm{Aut}_{B\cap K(A)}(B)$.
By Schinzel $B \cap K(A) = A$, so f is an isomorphism.

Theorem 1: $0 \to \mathrm{Gal}(K(B)/K) \to \mathrm{Aut}_{K^*}(B) \to E(B) \to 0$

Proof. Put $A = B_{\mathrm{ab}}$, then we have an exact sequence

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Corollary. The restriction map $E(B) \to E(B_{\mathrm{ab}})$ is an isomorphism.

For a radical group B over K which is Galois, we put

$$K\{B\} = K[B] \otimes_{K[K^*]} K$$
$$K\langle B\rangle = S^{-1}K\{B\},\ S = \langle x-1 :\ x \in B, x \neq 1\rangle$$

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If B is Galois, then $K\langle B\rangle$ is a Galois algebra with group $\mathrm{Aut}_{K^*}(B)$. Its invariants $K\langle B\rangle_{\mathrm{spl}}$ under $\mathrm{Gal}(K(\sigma B)/K)$ form a split K-algebra, whose components form an $E(B)$-torsor.

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Example. For $B = \langle \mathbb{Q}^*, \sqrt{-3}, \zeta\rangle$ with $\zeta = \sqrt[3]{1}$ we have $\mathbb{Q}\langle B\rangle_{\mathrm{spl}} = \mathbb{Q} \oplus x\mathbb{Q}$ where $x = (\zeta - \zeta^2)\sqrt{-3}$ satisfies $x^2 = 9$.

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Note. Even if B is not Galois, B_{ab} is Galois, and the components of $K\langle B\rangle$ are an $E(B_{\mathrm{ab}})$-torsor.

Let B be a radical group over K, let $\mu \subset B_{\mathrm{tor}}$ and let $W \subset B$ be a radical group over K which is Kummer, i.e., for all $x \in W$ there is a $w \geq 1$ with $x^w \in K$ and $\mu_w \subset K^*$.

Theorem 2. We have a natural isomorphism

$$E(\mu W) \to \mathrm{Gal}(\mathbb{Q}(\mu) \cap K(W)/\mathbb{Q}(\mu \cap W)).$$

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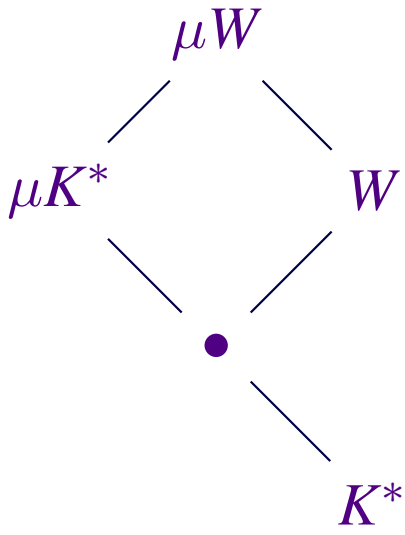
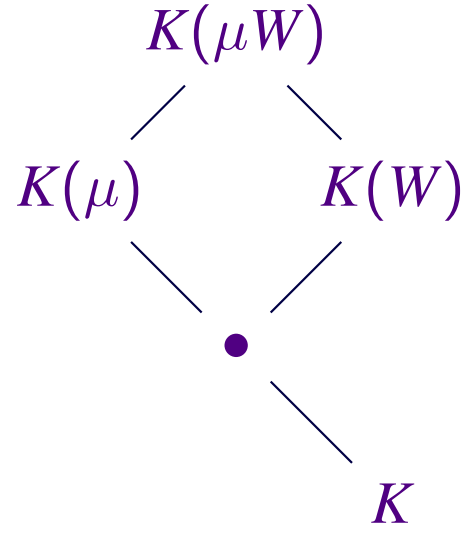
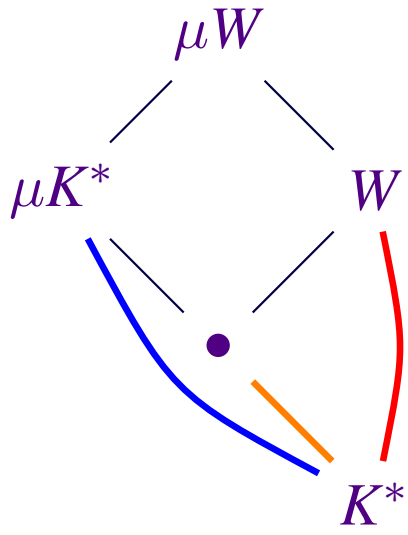
Taking $K = \mathbb{Q}$ and $W = \langle \sqrt[2]{p} : \ p \text{ prime} \rangle$, and μ all roots of unity, we find that $(\sqrt[\infty]{\mathbb{Q}^*})_{\text{ab}} = \mu W$. Combining the two theorems:

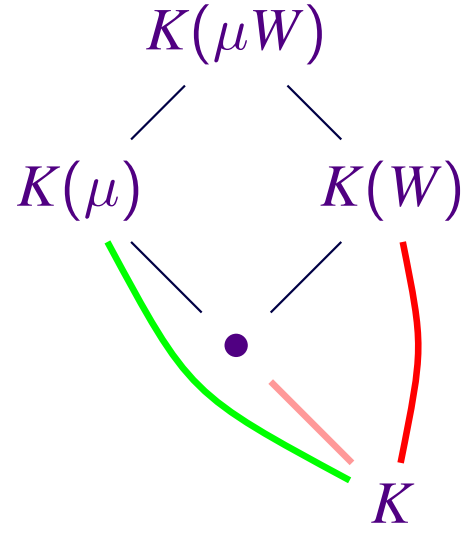
$$\mathsf{E}_{\mathbb{Q}} = E(\sqrt[\infty]{\mathbb{Q}^*}) = \mathrm{Gal}(\mathbb{Q}(W)/\mathbb{Q}) = \{1, -1\}^{\{\text{primes}\}}$$


μW
μK^*
W
K^*


$K(\mu W)$
$K(\mu)$
$K(W)$
K

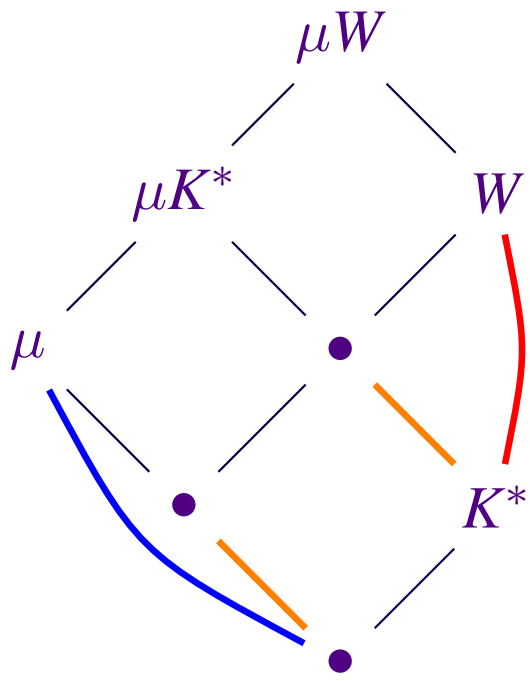
» Proof of Theorem 2 «

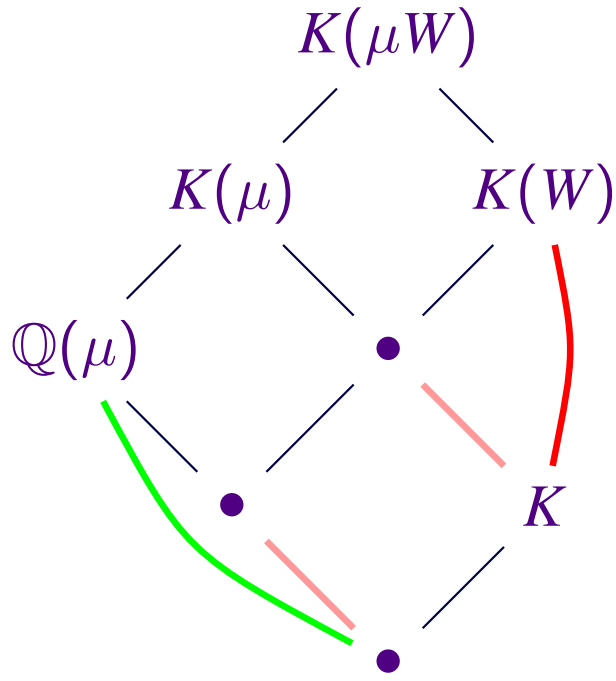




$\mathrm{Aut}_{K^*}(\mu W) =$ ■ $\times$ ■ ■

$\mathrm{Gal}(K(\mu W)/K) =$ ■ $\times$ ■ ■





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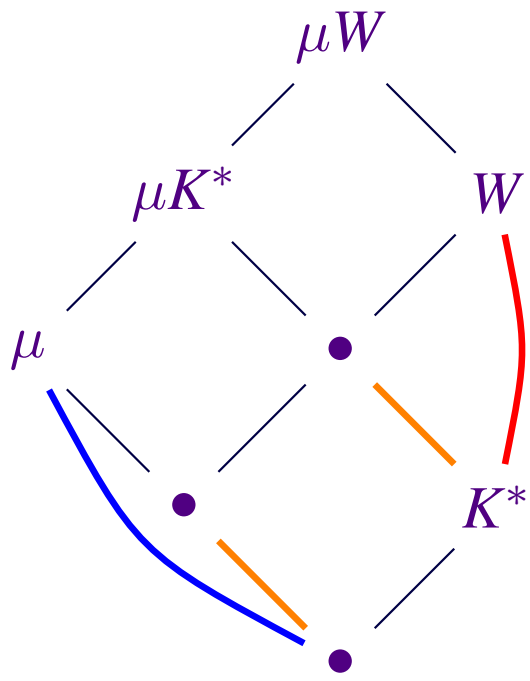

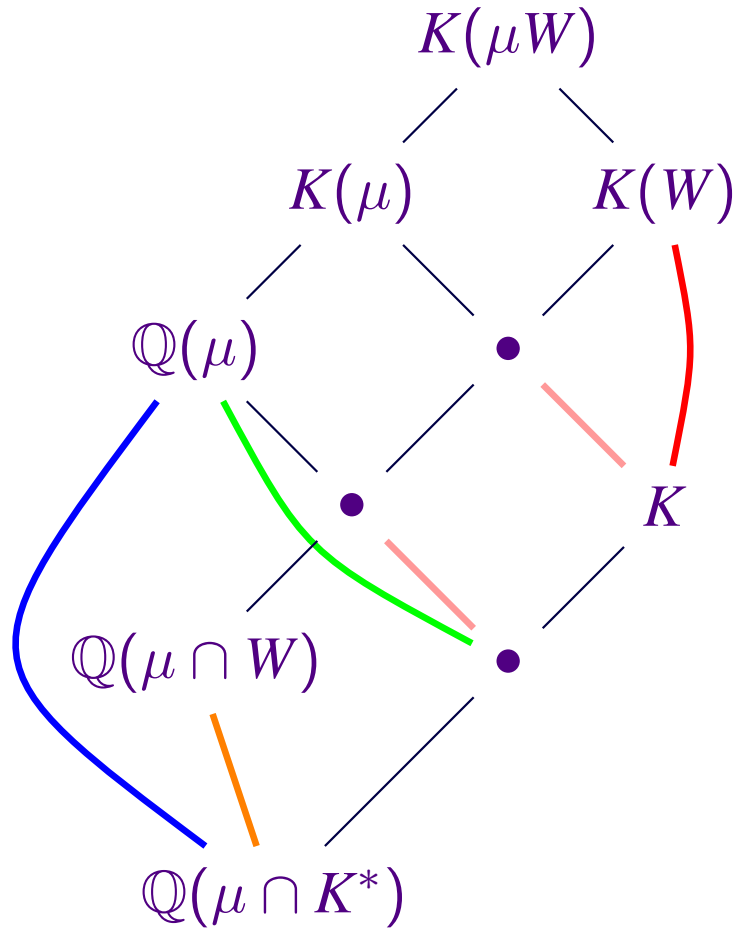

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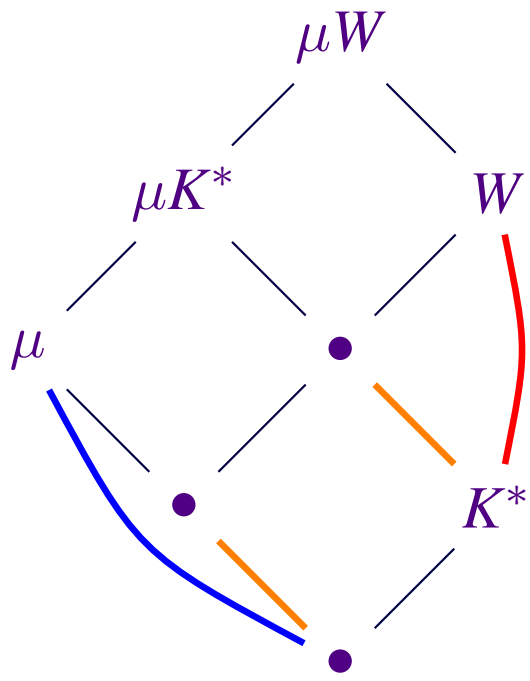

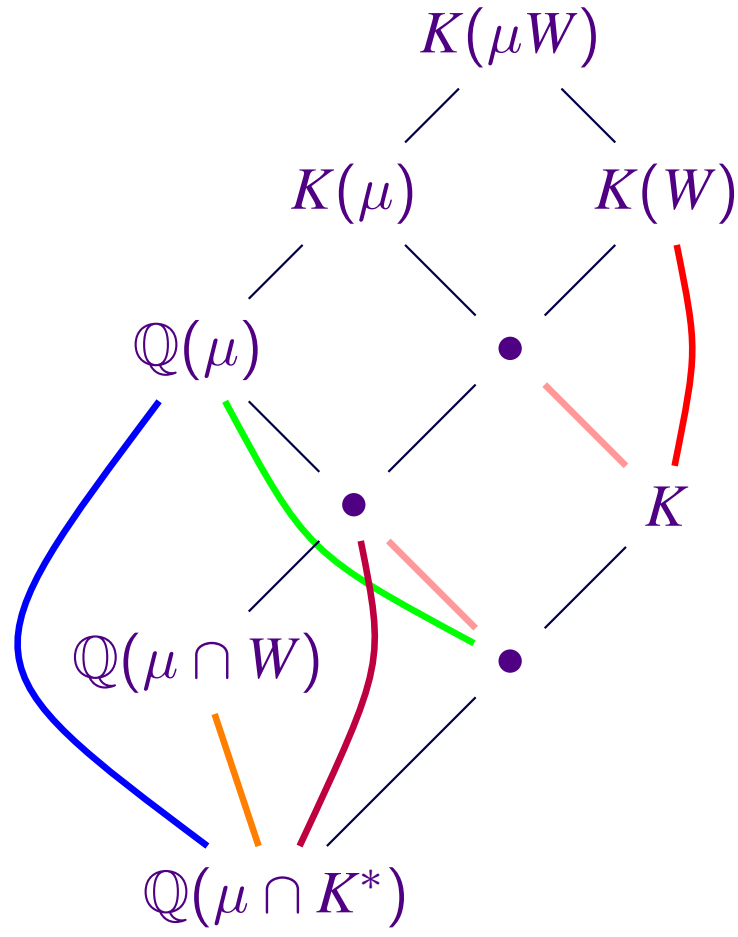

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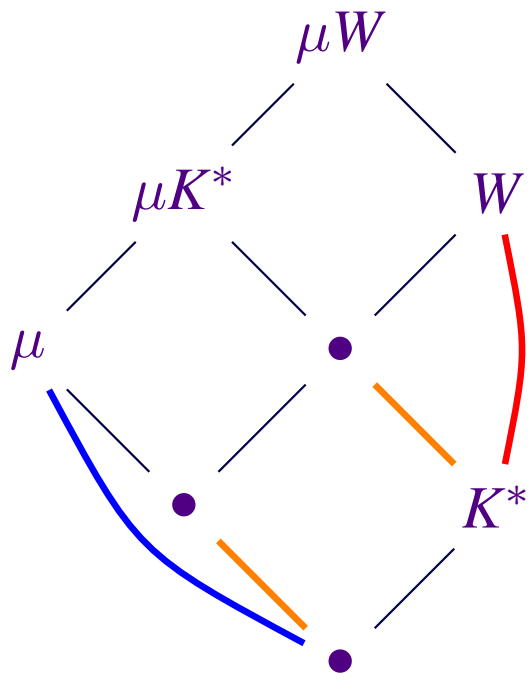

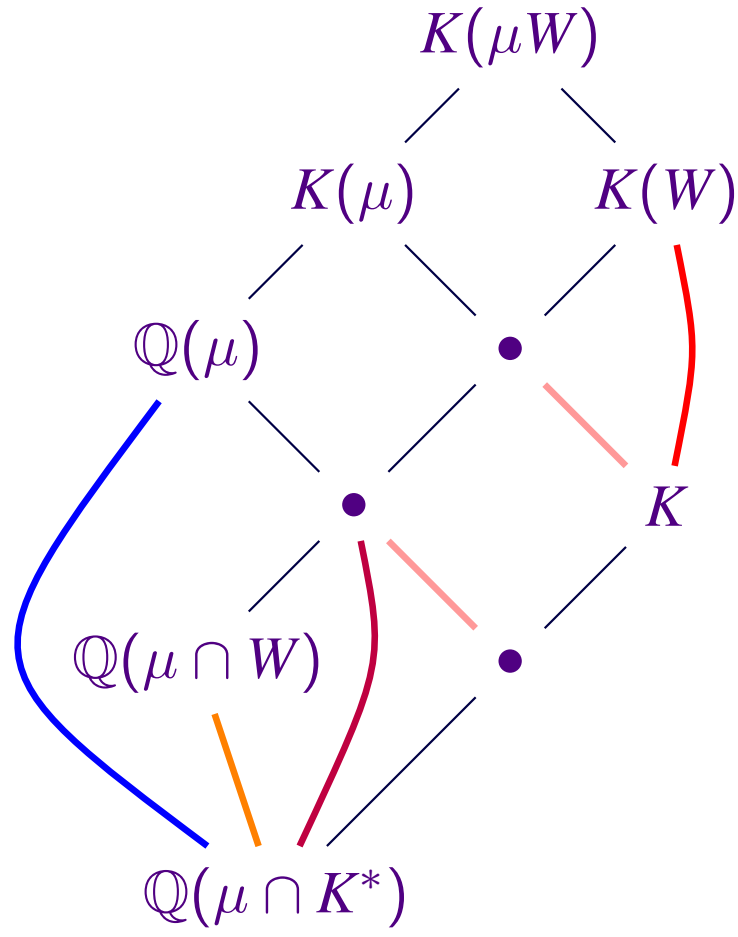

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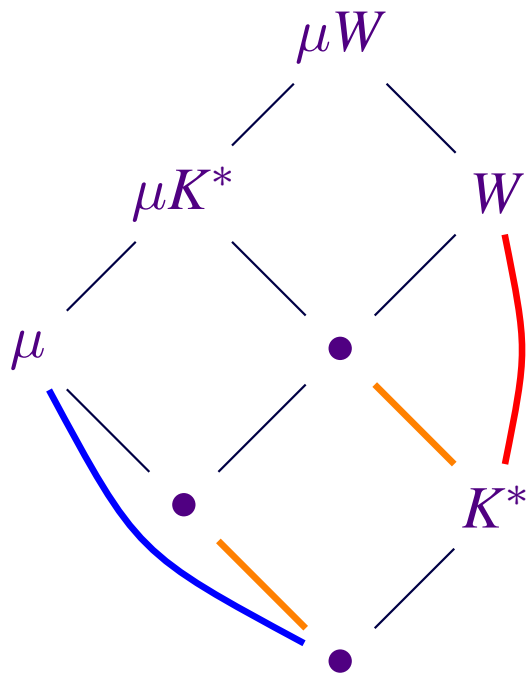

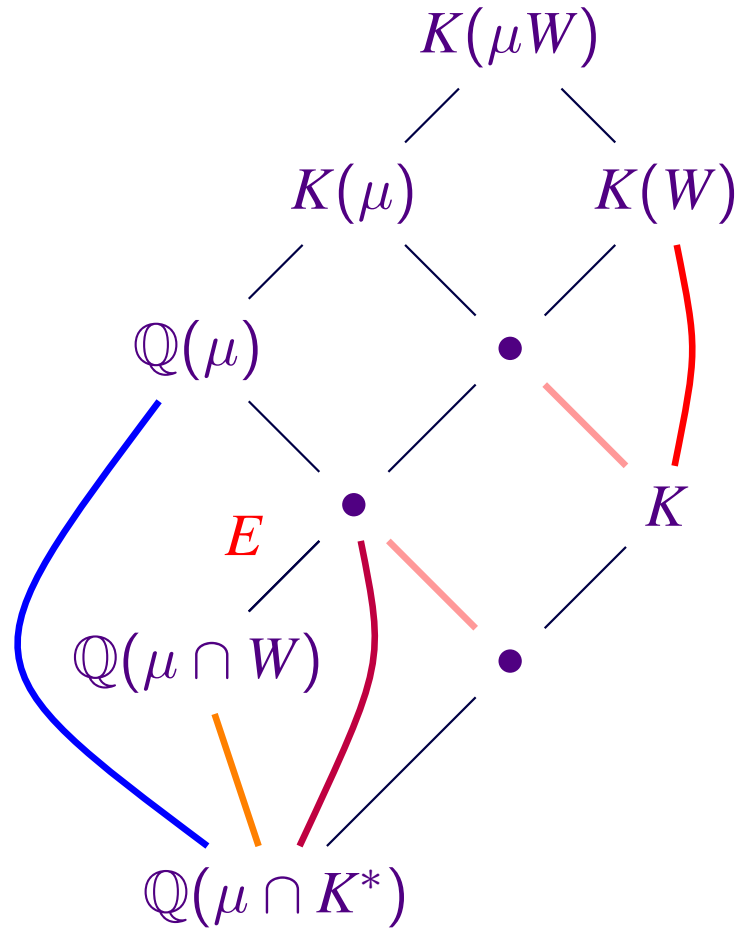

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For applications one often needs to understand the map

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$$\mathrm{Aut}_{K^*}(\mu W) \to E(\mu W) \cong \mathrm{Gal}(\mathbb{Q}(\mu) \cap K(W)/\mathbb{Q}(\mu \cap W)).$$

It is the difference of two restriction maps:

$$\mathrm{Aut}_{K^*}(\mu W) \to \mathrm{Aut}(\mu) \cong \mathrm{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\mu) \cap K(W)/\mathbb{Q}).$$

and

$$\mathrm{Aut}_{K^*}(\mu W) \to \mathrm{Aut}_{K^*}(W) \cong \mathrm{Gal}(K(W)/K) \to \mathrm{Gal}(K(W) \cap \mathbb{Q}(\mu)/\mathbb{Q}).$$

Suppose that $a \in \mathbb{Q}$ so that a and $-a$ are not squares. Let

$$B = \langle \mathbb{Q}^*, \sqrt[4]{-a^2} \rangle = B_{\mathsf{ab}}$$

Note that $\sqrt{-1} \in B$, so B is Galois.

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$$\begin{aligned} E(\mu_8 W) &= \mathsf{Gal}(\mathbb{Q}(\mu_8) \cap \mathbb{Q}(\sqrt{a})/\mathbb{Q}(\mu_8 \cap W)) \\ &= 0 \text{ if } 2a, -2a \notin \mathbb{Q}^{*2} \\ &\cong \mathbb{Z}/2\mathbb{Z} \text{ otherwise} \end{aligned}$$

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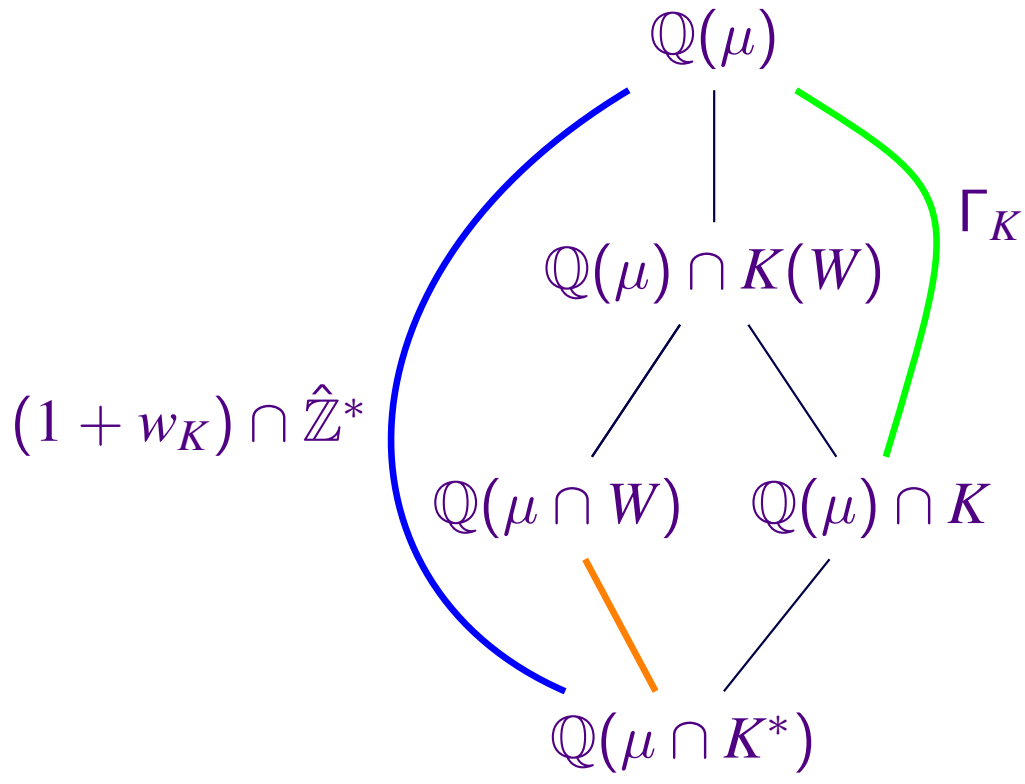
We can now compute $E(B)$ from the exact sequence

$$\mathsf{Aut}_B(\mu_8 W) \to E(\mu_8 W) \to E(B) \to 0$$

Goal: “Compute” $E(\sqrt[\infty]{K^*})$.

Let Γ_K be the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Aut}(\mu) = \hat{\mathbb{Z}}^*$.
Let $w_K = \mathrm{Ann}_{\hat{\mathbb{Z}}}(K^*_{\mathrm{tor}})$, let $\mu = \sqrt[\infty]{K^*}_{\mathrm{tor}}$.
Let $W = \{\text{Kummer radicals}\} = \{x \in \sqrt[\infty]{K^*} : \ (x \bmod K^*)^{w_K} = 1\}$.



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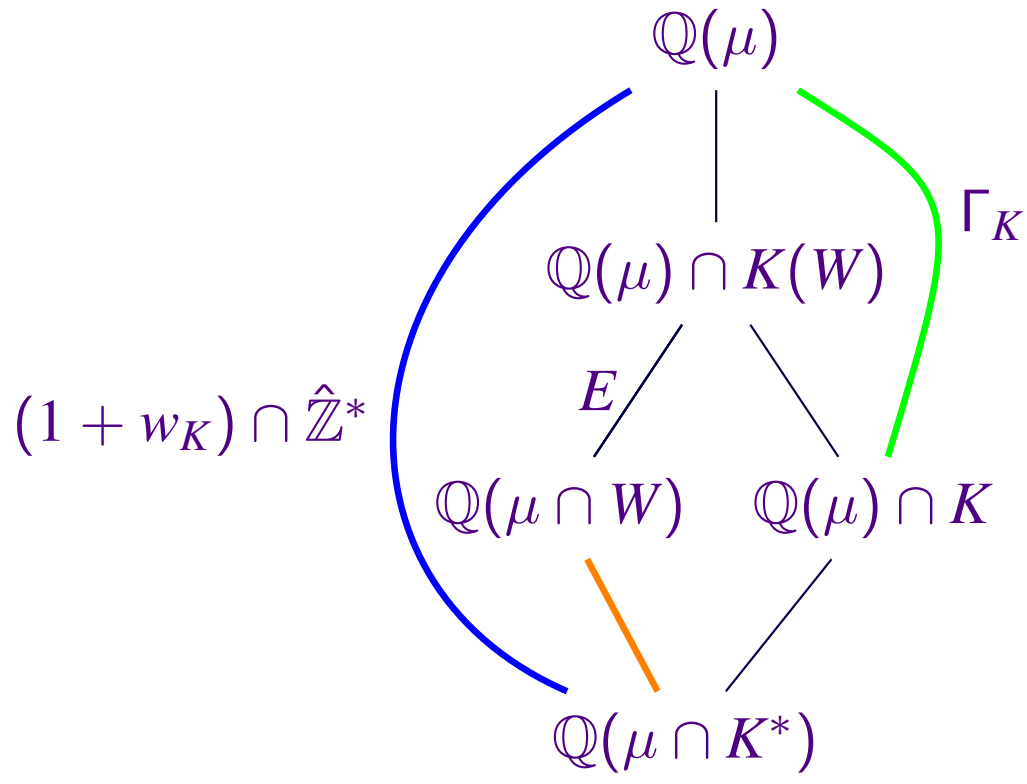

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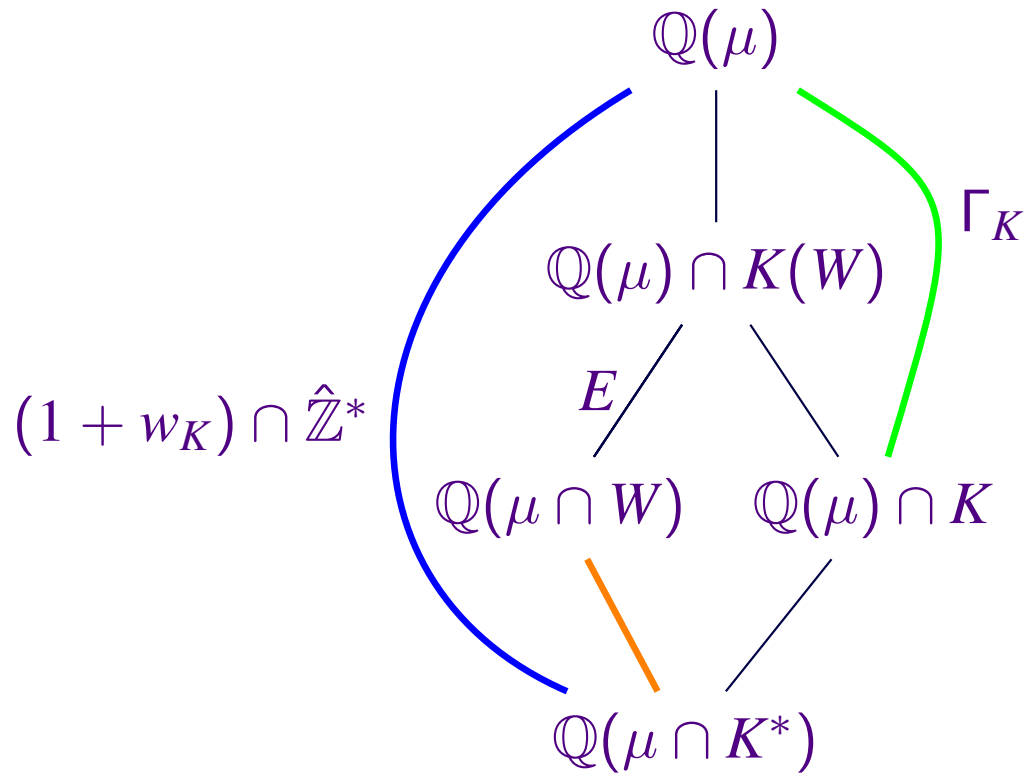

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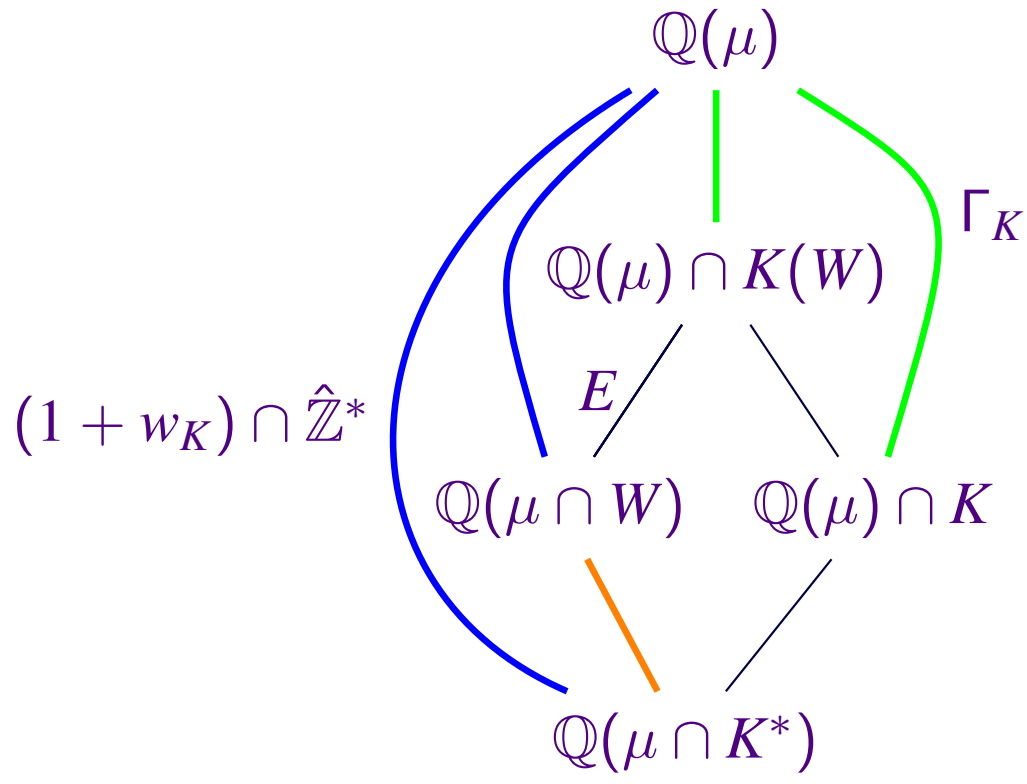

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$$\cong (1 + w_K^2) \cap \hat{\mathbb{Z}}^* / \Gamma_K^{w_K}$$